10-DIM EINSTEIN SPACES MADE UP ON BASIS OF 6-DIM RICCI- FLAT SPACES AND 4-DIM EINSTEIN SPACES

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Abstract

Some examples of ten-dimensional vacuum Einstein spaces (${}^{10}R_{ij} = 0$) made up on basis of four-dimensional Ricci-flat (${}^{4}R_{ij} = 0$) Einstein spaces and six-dimensional Ricci-flat spaces (${}^{6}R_{ij} = 0$) defined by solutions of the classical $Sin - Gordon$ equation are constructed.

The properties of geodesics for such type of the spaces are discussed.

1 Introduction

The properties of classical four dimensional Einstein spaces are dependent on the energy-momentum tensor of matter T_{ik}

$$R_{ij} = \frac{8\pi\kappa}{c^4}\left(T_{ik} - \frac{1}{2}g_{ik}T\right). \tag{1}$$

Tensor T_{ik} is self-dependent object in the Einstein theory of gravitation and in general does not has geometric description.

The most popular approach to the geometric description of the matter tensor T_{ik} and their relation with a Ricci-flat tensor ${}^{4}R_{ij}$ of Space-Time takes place within the bounds of the Kaluza-Klein theories using the string theory on Calaby-Yau manifolds.

At the same time both substance - Space-Time and the Matter are considered as a single whole.

We present here a new possibilities for such type of considerations.

2 Three-dimensional space of constant negative curvature

We start from a three-dimensional space endowed with the metrics in form

$$ds^2 = dx^2 + 2\cos(u(x,y)dxdy + dy^2 + A(x,y)^2 dz^2. \tag{2}$$

The condition on the space to be the space of constant negative curvature

$$R_{ijkl} - \lambda\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) = 0$$

in the case $\lambda = -1$ lead to the compatible system of equations

$$\frac{\partial^2}{\partial x \partial y}u(x,y) - \sin(u(x,y)) = 0,$$

$$\frac{\partial^2}{\partial x \partial y}A(x,y) - A(x,y)\cos(u(x,y)) = 0,$$

$$\frac{\partial^2}{\partial x^2}A(x,y) - \frac{\cos(u(x,y))\left(\frac{\partial}{\partial x}u(x,y)\right)\frac{\partial}{\partial x}A(x,y)}{\sin(u(x,y))} - A(x,y) + \frac{\left(\frac{\partial}{\partial x}u(x,y)\right)\frac{\partial}{\partial y}A(x,y)}{\sin(u(x,y))} = 0,$$

$$\frac{\partial^2}{\partial y^2}A(x,y) - \frac{\cos(u(x,y))\left(\frac{\partial}{\partial y}u(x,y)\right)\frac{\partial}{\partial y}A(x,y)}{\sin(u(x,y))} - A(x,y) + \frac{\left(\frac{\partial}{\partial x}A(x,y)\right)\frac{\partial}{\partial y}u(x,y)}{\sin(u(x,y))} = 0. \qquad (3)$$

As it follows that for any solution $u(x,y)$ of the $Sin-Gordon$ equation one possible to find the function $A(x,y)$ by solving corresponding linear system of equations.

So the following theorem is valid ([1]).

Theorem 1 *Three-dimensional spaces having the metric (2) with the functions $A(x,y)$ and $u(x,y)$ defined by a given system of equations are the spaces of constant negative curvature $\lambda = -1$.*

To take one example.

The simplest solution of the equation

$$\frac{\partial^2}{\partial x \partial y}u(x,y) - \sin(u(x,y)) = 0$$

is given by

$$u(x,y) = 4\arctan(\exp(x+y)).$$

At this condition the linear system looks as

$$\frac{\partial^2}{\partial x \partial y}A(x,y) + \frac{\left(6\,e^{2\,x+2\,y} - 1 - e^{4\,x+4\,y}\right)A(x,y)}{\left(1+e^{2\,x+2\,y}\right)^2} = 0,$$

$$\left(1-e^{4\,x+4\,y}\right)A(x,y)+\left(-2\,e^{2\,x+2\,y}-e^{4\,x+4\,y}-1\right)\frac{\partial}{\partial y}A(x,y)+\left(-6\,e^{2\,x+2\,y}+1+e^{4\,x+4\,y}\right)\frac{\partial}{\partial x}A(x,y)+$$

$$+\left(-1+e^{4\,x+4\,y}\right)\frac{\partial^2}{\partial x^2}A(x,y) = 0,$$

$$\left(1-e^{4\,x+4\,y}\right)A(x,y)+\left(-6\,e^{2\,x+2\,y}+1+e^{4\,x+4\,y}\right)\frac{\partial}{\partial y}A(x,y)+\left(-2\,e^{2\,x+2\,y}-e^{4\,x+4\,y}-1\right)\frac{\partial}{\partial x}A(x,y)+$$

$$+\left(-1+e^{4\,x+4\,y}\right)\frac{\partial^2}{\partial y^2}A(x,y) = 0$$

The simplest solution of this system is

$$A(x,y) = \frac{e^{x+y}}{1+e^{2\,x+2\,y}}.$$

In result we get the example of the metric of constant negative curvature

$$ds^2 = dx^2 + 2\cos(4\arctan(\exp(x+y)))dxdy + dy^2 + \left(\frac{\exp(x+y)}{1+\exp(2x+2y)}\right)^2 dz^2. \qquad (4)$$

Remark 1 *In 3-dimensional geometry the density of Chern-Simons invariant defined by*

$$CS(\Gamma) = \epsilon^{ijk}(\Gamma^p_{iq}\Gamma^q_{kp;j} + \frac{2}{3}\Gamma^p_{iq}\Gamma^q_{jr}\Gamma^r_{kp}) \qquad (5)$$

has an important role.

For the metric (2) one get

$$CS(\Gamma) = 0.$$

Our construction of ten dimensional Einstein space be composed from a few steps.

The first one is the creation of the six dimensional basic space with a necessary properties.

With this aim we use the notion of the Riemann extension of a given three-dimensional space.

3 Six dimensional Riemann extensions of the metrics of constant negative curvature

The Riemann extension of riemannian or nonriemannian spaces can be constructed with the help of the Chistoffel coefficients Γ^i_{jk} of corresponding Riemann space or with the help of connection coefficients Π^i_{jk} of affinely connected space.

The metrics of the Riemann extension of a given n-dimensional riemannian space looks as

$$ {}^{2n}ds^2 = -2\Gamma^i_{jk}dx^j dx^k \psi_i + 2dx^i d\psi_i, \tag{6}$$

where ψ_i are some additional coordinates.

To give an examples of the properties of Riemann extensions.

Theorem 2 *The Riemann extension of 3-dimensional space of constant curvature is a six-dimensional symmetrical space*

$$ {}^6R_{ijkl;m} = 0.$$

Theorem 3 *The Riemann extension of Ricci-flat ${}^nR_{ik} = 0$ space is a Ricci-flat ${}^{2n}R_{ik} = 0$ space.*

Theorem 4 *The spaces with conditions*

$$R_{ij;k} + R_{ki;j} + R_{jk;i} = 0.$$

on the Ricci-tensor conserve such conditions after the Riemann extension.

After the Riemann extension of three-dimensional space one get a six-dimensional space having the signature $[+ + + - - -]$

In case of the metric (2) we have a following components of the Christoffel symbols

$$\Gamma^2_{33} = \frac{A(x,y)\cos(u(x,y))\frac{\partial}{\partial x}A(x,y)}{(\sin(u(x,y)))^2} - \frac{A(x,y)\frac{\partial}{\partial y}A(x,y)}{(\sin(u(x,y)))^2},$$

$$\Gamma^1_{33} = -\frac{A(x,y)\frac{\partial}{\partial x}A(x,y)}{(\sin(u(x,y)))^2} + \frac{A(x,y)\cos(u(x,y))\frac{\partial}{\partial y}A(x,y)}{(\sin(u(x,y)))^2},$$

$$\Gamma^3_{23} = \frac{\frac{\partial}{\partial y}A(x,y)}{A(x,y)}, \quad \Gamma^2_{22} = \frac{\cos(u(x,y))\frac{\partial}{\partial y}u(x,y)}{\sin(u(x,y))}, \quad \Gamma^1_{22} = -\frac{\frac{\partial}{\partial y}u(x,y)}{\sin(u(x,y))},$$

$$\Gamma^3_{13} = \frac{\frac{\partial}{\partial x}A(x,y)}{A(x,y)}, \quad \Gamma^2_{11} = -\frac{\frac{\partial}{\partial x}u(x,y)}{\sin(u(x,y))}, \quad \Gamma^1_{11} = \frac{\cos(u(x,y))\frac{\partial}{\partial x}u(x,y)}{\sin(u(x,y))}.$$

So using these values we get from the (6) the metrics of six-dimensional extension

$$ {}^6ds^2 = \left(2\,\frac{\left(\frac{\partial}{\partial x}u(x,y)\right)V}{\sin(u(x,y))} - 2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial x}u(x,y)\right)U}{\sin(u(x,y))}\right)dx^2 - 4\,\frac{\left(\frac{\partial}{\partial x}A(x,y)\right)W\,dx\,dz}{A(x,y)} +$$

$$+2\,dx\,dU + \left(-2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial y}u(x,y)\right)V}{\sin(u(x,y))} + 2\,\frac{\left(\frac{\partial}{\partial y}u(x,y)\right)U}{\sin(u(x,y))}\right)dy^2 - 4\,\frac{\left(\frac{\partial}{\partial y}A(x,y)\right)W\,dy\,dz}{A(x,y)} + 2\,dy\,dV +$$

$$+A(x,y)\left(-2\,\frac{V\cos(u(x,y))\frac{\partial}{\partial x}A(x,y)}{(\sin(u(x,y)))^2} + 2\,\frac{V\frac{\partial}{\partial y}A(x,y)}{(\sin(u(x,y)))^2} + 2\,\frac{U\frac{\partial}{\partial x}A(x,y)}{(\sin(u(x,y)))^2} - 2\,\frac{U\cos(u(x,y))\frac{\partial}{\partial y}A(x,y)}{(\sin(u(x,y)))^2}\right)dz^2 +$$

$$+2\,dz\,dW, \tag{7}$$

where U, V, W are the set of additional coordinates.

Remark that we have the relations between the set of solutions of the system (3) and corresponding six-dimensional spaces endowed with the metrics (7). In particular we get the set of such type of the spaces corresponded the soliton solutions of the $Sin - Gordon$-equation.

Ricci-tensor of the metric (7) has non zero components ${}^6R_{ik} \neq 0$ and the next problem is transformation of the space (7) into the Ricci flat ${}^6R_{ij} = 0$ space defined by the solutions of the system (3).

4 Six-dimensional Ricci-flat space

For construction of the Ricci-flat six-dimensional space we consider the metrics which are conformal to the metrics (7)

$$ {}^6d\tilde{s}^2 = \frac{{}^6ds^2}{A(x,y)^2}, \tag{8}$$

with the function $A(x, y)$ defined by the solutions of the system (3).

The calculation with a GRTensorII ([12]) show that the Ricci-tensor of the metrics (8) has only one non zero component

$$R_{zz} \neq 0.$$

In explicit form it looks as

$$R_{zz} = -\left(\cos(u(x,y))\right)^2 \left(A(x,y)\right)^2 + 2\cos(u(x,y))\left(\frac{\partial}{\partial y}A(x,y)\right)\frac{\partial}{\partial x}A(x,y) + $$
$$ + \left(A(x,y)\right)^2 - \left(\frac{\partial}{\partial x}A(x,y)\right)^2 - \left(\frac{\partial}{\partial y}A(x,y)\right)^2 \tag{9}$$

It is interested to note that this quantity is an integral of the system (3).

In fact after differentiation the quantity (9) on the variables x or y we get the expressions containing the second order derivatives of the functions A_{xx}, A_{xy}, A_{yy}. The substitution of the corresponding values from the system (3) make these expressions vanish.

Remark 2 *The meaning of the value R_{zz} is dependent from the choice of the solutions of the system (3).*

For example on the solution

$$u(x,y) = 4\arctan(\exp(x+y)), \quad A(x,y) = \frac{\exp(x+y)}{1+\exp(2x+2y)}$$

we get

$$R_{zz} = 0.$$

At the same time on the singular solution of the system (3)

$$u(x,y) = 4\,\arctan(e^{i(x-y)}), \quad A(x,y) = \frac{e^{i(x-y)}}{1+e^{2\,i(x-y)}}$$

we get

$$R_{zz} = -4\,\frac{e^{2\,i(3\,y+x)} - 2\,e^{4\,i(y+x)} + e^{2\,i(y+3\,x)}}{\left(e^{2\,iy} + e^{2\,ix}\right)^4}.$$

So we get the family of six-dimensional metrics (8) depending from the solutions of the $Sin-Gordon$-equation and having only one non zero component of the Ricci-tensor.

Now for achievement of our aim we must add some additional terms into the expressions for the metrics (8) in such a way that the Ricci-flat six-dimensional manifold defined by the solutions of the $Sin-Gordon$-equation will be obtained.

Remark that there are a lot possibilities to make that.

All of them are connected with the freedom in choice of connections coefficients at the construction of our six-dimensional manifold in result of the extensions of concrete three-dimensional space(there are only eight components of the connections in the expression (8)).

For example adding the term

$$\frac{F(x,y)}{A(x,y)^2}U$$

into the component g_{zz} of metric tensor g_{ij} give rise to the metric

$$\begin{gathered}
{}^6ds^2 = \left(2\,\frac{\left(\frac{\partial}{\partial x}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} - 2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial x}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dx^2 - 4\,\frac{\left(\frac{\partial}{\partial x}A(x,y)\right)W\,dx\,dz}{(A(x,y))^3} + \\
+2\,\frac{dx\,dU}{(A(x,y))^2} + \left(-2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial y}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} + 2\,\frac{\left(\frac{\partial}{\partial y}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dy^2 - 4\,\frac{\left(\frac{\partial}{\partial y}A(x,y)\right)W\,dy\,dz}{(A(x,y))^3} + \\
+2\,\frac{dy\,dV}{(A(x,y))^2} + \\
+\left(-2\,\frac{\cos(u(x,y))V\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + 2\,\frac{V\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + \frac{F(x,y)U}{(A(x,y))^2} + 2\,\frac{U\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2}\right)dz^2 - \\
\left(-2\,\frac{\cos(u(x,y))U\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2}\right)dz^2 + 2\,\frac{dz\,dW}{(A(x,y))^2}.
\end{gathered} \tag{10}$$

This metric is a Ricci-flat

$${}^6R_{ij} = 0$$

but non a flat! $R_{ijkl} \neq 0$ if the function $F(x,y)$ is satisfied the equation

$$\begin{gathered}
\frac{\partial}{\partial x}F(x,y) - \frac{-\left(\frac{\partial}{\partial x}u(x,y)\right)A(x,y)\cos(u(x,y)) + 3\,\left(\frac{\partial}{\partial x}A(x,y)\right)\sin(u(x,y))}{A(x,y)\sin(u(x,y))} + \\
+4\,\frac{(A(x,y))^2\left(\cos(u(x,y))\right)^2 - 2\,\cos(u(x,y))\left(\frac{\partial}{\partial y}A(x,y)\right)\frac{\partial}{\partial x}A(x,y) - (A(x,y))^2 + \left(\frac{\partial}{\partial y}A(x,y)\right)^2 + \left(\frac{\partial}{\partial x}A(x,y)\right)^2}{-1 + \left(\cos(u(x,y))\right)^2} = \\
= 0.
\end{gathered} \tag{11}$$

In result we get the set of Ricci-flat six-dimensional manifolds defined by the solutions of the $Sin-Gordon$-equation.

According the geometric approach such type of manifold may be suitable arena for the string theory.

5 3 × 2+4=10

Now we present the construction of ten-dimensional Ricci-flat space

$$ {}^{10}R_{ik} = 0 $$

made up on six-dimensional Ricci-flat manifold and four-dimensional Ricci-flat Einstein space.

By way as example of four-dim Einstein space will be considered the Schwarzschild Space-Time with the metric

$$ ds^2 = (1 - M/r)c^2dt^2 - r^2(\sin(\theta)^2 d\phi^2 + d\theta^2) - dr^2/(1 - M/r). $$

As the six-dimensional Ricci-flat space will be used the space with the metric (8) with a suitable additional term.

$$ {}^{10}ds^2 = \left(2\,\frac{\left(\frac{\partial}{\partial x}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} - 2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial x}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dx^2 - 4\,\frac{\left(\frac{\partial}{\partial x}A(x,y)\right)W\,dx\,dz}{(A(x,y))^3} + 2\,\frac{dx\,dU}{(A(x,y))^2} + $$
$$ + \left(-2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial y}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} + 2\,\frac{\left(\frac{\partial}{\partial y}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dy^2 - 4\,\frac{\left(\frac{\partial}{\partial y}A(x,y)\right)W\,dy\,dz}{(A(x,y))^3} + 2\,\frac{dy\,dV}{(A(x,y))^2} + $$
$$ + \left(-2\,\frac{\cos(u(x,y))V\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + 2\,\frac{V\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + 2\,\frac{U\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} - 2\,\frac{\cos(u(x,y))U\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2}\right)dz^2 + $$
$$ +2\,\frac{dz\,dW}{(A(x,y))^2} - dr^2\left(1 - \frac{M}{r}\right)^{-1} - r^2\left(\sin(\theta)\right)^2(d(\phi))^2 - r^2\,(d(\theta))^2 + \left(c^2 - \frac{c^2M}{r}\right)dt^2 + $$
$$ + \frac{F(x,y)U}{(A(x,y))^2}dz^2. \tag{12} $$

In result we have the metric of the union space which is composed from two independent spaces.

In particular the geodesic equations of full 10D-space are dissolved on two independent subsystems of equations - for the coordinates (x, y, z, U, V, W) and (r, θ, ϕ, t).

Under such condition on the metric the influence of 6D-space which is the medium of the Matter (analogue of Calaby-Yau space!) on the properties of 4D-Space-Time is absent.

For description of interaction between both structures it is necessary to introduce additional terms into the expression for the metric.

As example we consider a following metric of 10D-space

$$ {}^{10}ds^2 = \left(2\,\frac{\left(\frac{\partial}{\partial x}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} - 2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial x}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dx^2 - 4\,\frac{\left(\frac{\partial}{\partial x}A(x,y)\right)W\,dx\,dz}{(A(x,y))^3} + 2\,\frac{dx\,dU}{(A(x,y))^2} + $$
$$ + \left(-2\,\frac{\cos(u(x,y))\left(\frac{\partial}{\partial y}u(x,y)\right)V}{(A(x,y))^2\sin(u(x,y))} + 2\,\frac{\left(\frac{\partial}{\partial y}u(x,y)\right)U}{(A(x,y))^2\sin(u(x,y))}\right)dy^2 - 4\,\frac{\left(\frac{\partial}{\partial y}A(x,y)\right)W\,dy\,dz}{(A(x,y))^3} + 2\,\frac{dy\,dV}{(A(x,y))^2} + $$
$$ + \left(-2\,\frac{\cos(u(x,y))V\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + 2\,\frac{V\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} + 2\,\frac{U\frac{\partial}{\partial x}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2} - 2\,\frac{\cos(u(x,y))U\frac{\partial}{\partial y}A(x,y)}{A(x,y)\left(\sin(u(x,y))\right)^2}\right)dz^2 + $$
$$ +2\,\frac{dz\,dW}{(A(x,y))^2} - dr^2\left(1 - \frac{M}{r}\right)^{-1} - r^2\left(\sin(\theta)\right)^2(d(\phi))^2 - r^2\,(d(\theta))^2 + \left(c^2 - \frac{c^2M}{r}\right)dt^2 + $$
$$ + \frac{H(r,t)}{(A(x,y))^2} + \frac{F(x,y)U}{(A(x,y))^2}, \tag{13} $$

where the new additional term $\frac{H(r,t)}{(A(x,y))^2}$ was added into the expression for the metric (12).

Taking into consideration the relation (11) the conditions on the metric (13) to be a Ricci-flat lead to the following equation for determination of the function $H(r,t)$

$$-r^3 \frac{\partial^2}{\partial t^2} H(r,t) + rc^2 (r-M)^2 \frac{\partial^2}{\partial r^2} H(r,t) + c^2 (2r-M)(r-M) \frac{\partial}{\partial r} H(r,t) = 0. \tag{14}$$

Remark 3 *The solutions of the equation (14) can be presented in form*

$$H(r,t) = F_2(t) F_1(r)$$

where the functions $F_2(t)$ and $F_1(r)$ satisfy the equations

$$\frac{d^2}{dt^2} F_2(t) = \_c_1 F_2(t) c^2,$$

and

$$\frac{d^2}{dr^2} F_1(r) = \frac{r^2 F_1(r)\_c_1}{(r-M)^2} - \frac{\left(\frac{d}{dr} F_1(r)\right)(2r-M)}{(r-M)r}.$$

where $\_c_1$ is arbitrary parameter.

The first equation is elementary and the second equation is more complicated and equivalent the equation without the first derivative

$$\frac{d^2}{dr^2} F_3(r) - 1/4 \frac{F_3(r)\left(4r^4\_c_1 - M^2\right)}{r^2 (r-M)^2} = 0$$

where

$$F_1(r) = \frac{F_3(r)}{\sqrt{(r-M)r}}.$$

So the metric of ten-dimensional Ricci-flat spaces dependent from the solutions of $Sin-Gordon$-equation has been constructed and our problem is solved.

Let us discuss the properties of geodesic equations in this case.

6 On geodesic equations

For simplicity's sake we consider a following equations for geodesic of the metric (13)

$$\frac{d^2}{ds^2} r(s) = (r-M)(\sin(\theta))^2 \left(\frac{d}{ds}\phi(s)\right)^2 + (r-M)\left(\frac{d}{ds}\theta(s)\right)^2 - 1/2 \frac{(r-M)c^2 M \left(\frac{d}{ds}t(s)\right)^2}{r^3} +$$
$$+1/2 \frac{M\left(\frac{d}{ds}r(s)\right)^2}{(r-M)r} - 1/2 \frac{(r-M)\left(\frac{\partial}{\partial r}H(r,t)\right)\left(\frac{d}{ds}z(s)\right)^2}{r(A(x,y))^2},$$

and

$$\frac{d^2}{ds^2} t(s) = -\frac{M\left(\frac{d}{ds}r(s)\right)\frac{d}{ds}t(s)}{(r-M)r} + 1/2 \frac{r\left(\frac{\partial}{\partial t}H(r,t)\right)\left(\frac{d}{ds}z(s)\right)^2}{(r-M)c^2(A(x,y))^2}.$$

Comparison of these expressions with standard equations for geodesics of the Schwarzshild Space-Time show us essential distinctions between them.

In fact we observe that the expressions for geodesics of general ten-dimensional Ricci-flat space are contained additional terms depending from the function $A(x,y)$ which in turn is dependent from solutions of the $Sin-Gordon$-equation arising in context of the theory of the spaces of constant negative curvature and corresponding Ricci-flat six-dimensional space (see (3)).

From a given point of view the reason of appearance of new terms in geodesic equations is pure geometric and is motivated by consideration of the problem in the spirit of the Kaluza-Klein theories.

At the same time six-dimensional Ricci-flat space defined by solutions of the $Sin - Gordon$ equation stands in the role of Calaby -Yau variety.

This fact offer the new challenge for the problem of geometrical description of joint properties of Space-Time and Matter.

Remark 4 *In the articles of author ([9]-[10]) analogous approache to the problem of description of joint properties of Space-Time and Matter was considered on the basis of the Korteveg-de Vrize (KdV) equations (Cylindrical KdV, mKdV) which are described some classes of three - dimensional metrics of zero curvature ([1]).*

Six-dimensional Ricci-flat Riemann extensions of 3-dimensional metrics depending from solutions of KdF-equations were constructed.

The properties of geodesics of ten-dimensional Ricci-flat spaces combined on basis of interactive six-dimensional Ricci-flat spaces and Schwarzschild or E.Kasner Space-Time were investigated.

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